NON-EUCLIDEAN INCIDENCE PLANES

BY

R. ARTZY*

ABSTRACT

The paper discusses finite or infinite incidence planes in which an oval plays the role of a metric conic. The points of the oval are used as coordinates, and ordered couples of these coordinates give rise to a coordinatization of the whole plane by means of ternary structures. These ternaries are studied, and a few specializations and their geometric analogues are studied.

Introduction. Projective and affine incidence planes have been studied extensively. It is, therefore, reasonable also to consider non-euclidean planes based on incidence axioms alone. One way of doing this would be a restriction to the Bolyai-Lobachevsky plane, to the exclusion of the points on the metric conic and those exterior to it. Some aspects of such planes were discussed by L. M. Graves [2] and T. G. Ostrom [6]. Another approach replaced the metric conic and its polarity in the classical Cayley-Klein model by an oval and a polarity with respect to it in a projective incidence plane. This was done, for instance, by T. G. Ostrom [5] and in a series of papers by R. Baer culminating in [1], where the author proved that under his restrictive axioms no Bolyai Lobachevsky plane can be finite. Most of the investigations in these papers were directed at finite planes, and many of the combinatorial and number-theoretic arguments employed in them do not carry over to the infinite case.

This paper deals with non-euclidean planes π_C, that is, finite or infinite projective incidence planes π in which the role of the metric conic is played by an oval C defined by incidence properties alone. The lines of π_C are shown to correspond bijectively to the ordered pairs of points of C. The incidence relation is expressed in terms of a ternary operation on the points of C, and the result is an algebraic structure called a *ternary*, somewhat resembling M. Hall's ternary ring [3], but possessing also quadratic properties.

This enables us to obtain a coordinatization of π_C starting from C, thus generalizing the classical theory which proceeded as follows: In the real projective plane a nonsingular conic was designated as the metric conic. An addition and a multiplication of the points on the conic were defined as described, for instance, in [9, p. 232]. The points of the conic were the "ends" of the lines in the Cayley-Klein model of the Bolyai-Labachevsky plane consisting of the interior of the

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conic. D. Hilbert [4, Appendix III] developed an "end calculus" which was the exact analogue of the field of the points on the metric conic under the addition and multiplication mentioned above.

In our treatment we define addition and multiplication of points of C in terms of the ternary. These operations closely resemble the classical operations. The points of C are shown to form commutative loops under this addition and multiplication. Associativity of the additive or the multiplicative loop, respectively, is proved to correspond to the validity of two special Pascal properties with C substituted for the conic. Even with these two properties and with linearity [3], the ternary is not necessarily right- or left-distributive. This is significant because, with the ternary a finite field, π would become desarguesian and C a conic, in view of B. Segre's result [8]. We would then obtain the classical Cayley-Klein model. If the ternary is a field of characteristic $\neq 2$, the coordinatization developed in this paper turns out to be essentially the same as that used in Hilbert's "end calculus".

A question arising naturally concerns non-euclidean collineations, that is, those collineations of π_C which preserve C. They will be discussed in another paper.

1. Definitions.

We define a set of points and lines to be a *non-euclidean* plane π_C if it is a projective plane π, that is, satisfies the axioms I, δI, II, and if it contains a subset C of points ("oval") satisfying the axioms III.

I. For any two distinct points there is a unique line through both.

δI. For any two distinct lines there is a unique point lying on both.

II. There are 4 points no 3 of which colline.

δI is the dual of I. It is well known that the dual of II is a consequence of I, δI, and II.

III. There is a nonempty set C of points such that III1, III2, δIII1 and δIII2 hold.

III1. Each point P of C is on just one line which contains no other point of C. This line is called PP, the *tangent* at P. We denote $\delta C = \{PP \mid P \in C\}$.

III2. No 3 points of C colline.

δIII1. Each line of δC contains just one point which is not also on another tangent.

δIII2. No 3 lines of δC concur.

Obviously the axioms of the non-euclidean plane are self-dual. The set of all the points mentioned in δIII1 is C. For, by III1, each tangent has one point which is not on any other tangent, namely the point in C, and by δIII1 this is the only point. Thus, $\delta\delta C = C$.

It follows from III1 and δIII1 that a point is in C if and only if it lies on just one tangent. If it lies on no tangent, it is called *interior*, and if it lies on two distinct tangents, it is *exterior*. Dually, each line either belongs to δC, or it contains two points of C and is called a *secant*, or one of its points belong to C and it is a *stray*. The set of all interior points may be considered as a generalized Bolyai-Lobachevsky plane.

R. Baer [1] discussed non-euclidean planes for which he also postulated that every line through an interior point be a secant and that every point on a stray be exterior. These requirements are not a consequence of our axioms I, II and III. As a counter-example we observe that in $PG(2, 3)$ the conic $x_0^2-x_1^2-x_2^2 = 0$ obviously satisfies all requirements of C. The point (1, 0, 0) is interior, but $x_1 = x_2$ through (1, 0, 0) is not a secant. Moreover, $x_1 = x_2$ is a stray, but (1, 0, 0) on it is not exterior.

Baer showed that his planes were necessarily infinite. Our planes may be finite or infinite.

2. Coordinates for π_C.

PROPOSITION 1. *C has at least* 3 *non-collinear points.*

Proof. Axiom II implies that the number of lines through each point of π is at least 3. At least one point P of π is in C. Only one of the $\geqq 3$ lines through P is a tangent, and hence, by III1, each of the remaining lines through P has to contain another point of C. These two additional distinct points cannot colline with P, in view of III2.

We label all points of C. Three of them will be called $0, 1, \infty (0 \neq 1 \neq \infty \neq 0)$, which is possible in view of Proposition 1. We denote by $(p)\,(q)$ the line joining the points p and q, and by $j \times k$ the point of intersection of the lines j and k.

In order to shorten the notation we will use the following definition. If $p \in C$ and j is a line through p, then $j \times C$ will be p if $j = (p)\,(p)$, and it will be the second point of intersection of j and C if j is a secant.

Now consider a line j not through ∞. Let (Figure 1) $j \times (0)\,(\infty) = X$, $j \times (\infty)\,(\infty) = Y$, $X(1) \times C = x$, and $Y(0) \times C = y$. Then the points x and y of C are uniquely determined by j. Conversely, if $x \neq \infty$ and $y \neq \infty$ are points of C, then by $(x)\,(1) \times (0)\,(\infty) = X$, $(y)\,(0) \times (\infty)\,(\infty) = Y$, a line $j = XY$ is uniquely determined by x and y. If $\Sigma = C - \{\infty\}$, we have, therefore, a bijectivity between $\Sigma \times \Sigma$ and all the lines of π other than those through ∞. In particular, we write $j = [x, y]$ and call x and y the *line coordinates* of j.

We have yet to take care of the lines through ∞. Let k be such a line, and let $k = (n)\,(\infty)$, $n \in C$. If $Q = (1)\,(1) \times (0)\,(\infty)$, and if $Q(n) \times C = m$, we will write $k = [m]$. Obviously this is a bijectivity between all the points of C and the set of all lines through ∞. In particular, this makes $(\infty)\,(\infty) = [0]$ and $(0)\,(\infty) = [\infty]$.

If P is a point of π not on $(0)\,(\infty)$ and not in C, then for each $x \in \Sigma$ there is just one line through P having x as its first coordinate, because for

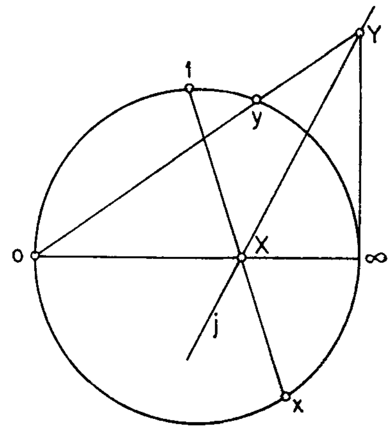

FIGURE 1

$X = (x)(1) \times (0)(\infty)$ the line XP is uniquely determined. If y is the second coordinate of this line, then y depends on x and on the choice of P. Let (Figure 2)

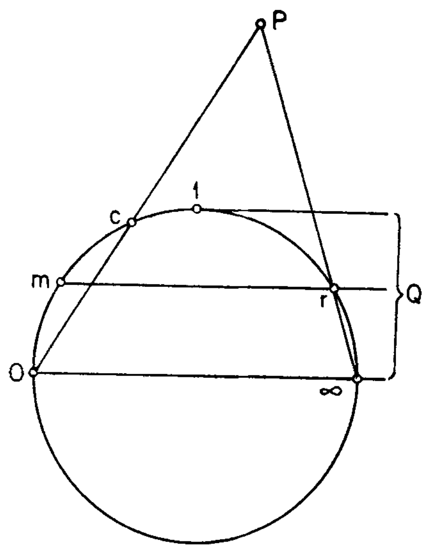

FIGURE 2

$P(0) \times C = c$, $P(\infty) \times C = r$, $(0)(\infty) \times (1)(1) = Q$, $Q(r) \times C = m$. Then P determines m and c uniquely, and conversely for each m and c from Σ there is a unique P. Thus y is a function of x, m, and c, and we write $y = T(x, m, c)$. This is the *equation of the point P*. T may be considered as a ternary operation $\Sigma \times \Sigma \times \Sigma \to \Sigma$.

Now let $P = p \in C$, but not on $(0)(\infty)$. Then we proceed as before, now putting $r = p = c$. Furthermore, if $P \neq 0$ lies on $(0)(\infty)$, and if $P(1) \times C = b$, then all the lines through P, except $P(\infty)$, will have the first coordinate b, and the equation of P will be $x = b$. Thus we obtained equations for all points of π, except ∞. So, finally, we define $x = \infty$ to be the equation of the point ∞.

3. **The ternary.** We now introduce an algebraic structure called a *ternary*, (S, T). The set S contains at least two distinct elements, 0 and 1, and is closed under the ternary operation T. Moreover, the axioms T1 through T12 hold in (S, T) for all a, b, c, and d in S. No claim is made as to the independence of the axioms.

T1. $T(0, b, c) = c$.

T2. $T(a, 0, c) = c$.

T3. The equation $T(x, b_1, c_1) = T(x, b_2, c_2)$, with $b_1 \neq b_2$, has a unique solution x in S.

T4. The equation $T(a, b, x) = d$ has a unique solution x in S.

T5. If $a_1 \neq a_2$, the simultaneous equations $T(a_1, x, y) = d_1$ and $T(a_2, x, y) = d_2$ have a solution for x and y in S.

PROPOSITION 2. *The solution in* T5 *is unique.*

Proof. Suppose there are two solutions x_1, y_1 and x_2, y_2. If $x_1 \neq x_2$, $T(a_1, x_1, y_1) = d_1 = T(a_1, x_2, y_2)$; then, by T3, a_1 is uniquely defined, and $a_1 = a_2$, a contradiction. If $x_1 = x_2$, then, by T4, also $y_1 = y_2$.

PROPOSITION 3. *If $b \neq 0$, $T(x, b, c) = d$ has a unique solution x.*

Proof. By T2, $d = T(x, 0, d)$, and by T3, $T(x, b, c) = T(x, 0, d)$ has a unique solution x.

PROPOSITION 4. *If $a \neq 0$, $T(a, x, c) = d$ has a unique solution x.*

Proof. By T1, $T(0, x, c) = c$. This equation, together with that of our statement, has a unique solution x, c by Proposition 2.

DEFINITION. For every $a \neq 0$, define a^{-1} by $T(a, 1, 1) = T(a, a^{-1}, a)$. The unique existence of a^{-1} follows from Proposition 4.

PROPOSITION 5. $1^{-1} = 1$.

T6. For all $a \neq 0$, $T(a, 1, 1) = T(a, x, a)$ implies $T(x, 1, 1) = T(x, a, x)$. This is equivalent to the statement $(a^{-1})^{-1} = a$.

PROPOSITION 6. *If $a \neq 0$, then $a^{-1} \neq 0$.*

Proof. If $a^{-1} = 0$, then, by T6, $T(0, 1, 1) = T(0, a, 0)$, which, by T1, means $1 = 0$, a contradiction.

PROPOSITION 7. *$a^{-1} = b^{-1}$ implies $a = b$.*

Proof. By T6, we have $T(b^{-1}, 1, 1) = T(b^{-1}, \mathrm{a}, b^{-1})$ and $T(b^{-1}, 1, 1) = T(b^{-1}, b, b^{-1})$. By Proposition 4, a $= b$.

DEFINITION. For $b \neq a \neq 0 \neq b$, we define $ab = ba$ and $a + b = b + \mathrm{a}$ by $T(ab, a^{-1}, a) = T(ab, b^{-1}, b) = a + b$.

Unique existence of $ab = ba$ follows from T3 and Proposition 7.

PROPOSITION 8. *$a1 = 1a = a$ if $1 \neq a \neq 0$.*

T7. $T(1, a^{-1}, a) = T(1, a, a^{-1})$ if $a \neq 0$.

PROPOSITION 9. *$aa^{-1} = 1$ if $1 \neq a \neq 0$ and if $a \neq a^{-1}$.*

T8. The equation $T(a, x^{-1}, x) = b$ has at most two solutions x in S.

PROPOSITION 10. *If $T(a, x^{-1}, x) = b$ has 2 distinct nonzero solutions x and y, then $x + y = b$ and $xy = a$.*

T9. For every nonzero a there are unique $p \neq 0$ and q in S such that $T(p, a^{-1}, a) = q$ and such that $T(p, b^{-1}, b) = q$ implies $a = b$.

DEFINITION. Under the assumptions of T9, $p = aa$ and $q = a + a$.

PROPOSITION 11. $1 \cdot 1 = 1$.

Proof. It is claimed that, in the terms of T9, $a = 1$ implies $p = 1$. Then from $T(1,1,1) = T(1, b^{-1}, b)$ it would have to follow that $b = 1$. Suppose $\mathrm{b} \neq 1$, then $b1 = 1$, by the definition of multiplication. But, by Proposition 8, $b1 = b$, a contradiction.

PROPOSITION 12. *If $a \neq 0 \neq d$, then there exists a unique x satisfying $ax = d$.*

Proof. By T8, $T(d, a^{-1}, a) = T(d, x^{-1}, x)$ has at most 2 solutions. One solution is $x = a$. If it is single, then by T9, $aa = d$. If there is a solution x other than a, then $ax = d$.

PROPOSITION 13. *If $e \neq a \neq 0$, then $a + x = e$ has a unique solution x.*

Proof. $T(ax, a^{-1}, a) = e$ has a unique solution ax, by Proposition 3. By Proposition 12 this yields a unique x.

DEFINITIONS. $a0 = 0a = 0$, $a + 0 = 0 + a = a$ for all a in S.

PROPOSITION 14. *If $a \neq 0$ and $ax = 0$, then $x = 0$.*

Proof. Suppose $x \neq 0$. If $x \neq a$, then $T(0, a^{-1}, a) = T(0, x^{-1}, x)$, that is, $x = a$, a contradiction. If $x = a$, then $ax = aa = p = 0$, which is impossible. Hence $x = 0$ is the only solution.

PROPOSITION 15. *$a + x = a$ has the only solution $x = 0$.*

Proof. Suppose $x \neq 0 \neq a \neq x$. Then $T(ax, a^{-1}, a) = a$. By Proposition 3, ax then must be 0, and by Proposition 14, $x = 0$, a contradiction. Now suppose $x = a \neq 0$. Then $T(p, a^{-1}, a) = a$, which yields the impossible value $p = 0$. Finally, let $a = 0$. Then $0 + x = 0$. But $0 + x = x$, and hence $x = 0$.

PROPOSITION 16. *$(S, +)$ and $(S - \{0\}, \cdot)$ are commutative loops.*

T10. $x + x = a$ has exactly one solution x.

PROPOSITION 17. *$b + b = 0$ implies $b = 0$.*

T11. $xx = a$ has the single solution $x = 0$ if $a = 0$, and no solutions or 2 distinct solutions in S otherwise.

T12. $T(xx, m, c) = x + x$ has the single solution $x = c$ if $m = c^{-1}$, and no solution or 2 distinct solutions in S if $0 \neq m \neq c^{-1}$.

4. The coordinate structure as ternary.

THEOREM 1. *(Σ, T), as defined in section 2, is a ternary satisfying* T1 *through* T12.

Proof. T1, T2, T4 and T5 are obviously satisfied. For T3 a unique point x will always be obtained unless $b_1 = b_2$, in which case x would be ∞. But ∞ is not in Σ. The construction of a^{-1} turns out to be (Figure 3):

$$((1)(1) \times (0)(\infty))(a) \times C = a^{-1},$$

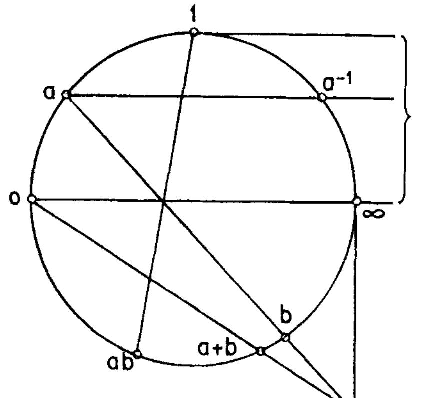

FIGURE 3

and then T6 and T7 follow immediately. The points of C except ∞ and 0 have exactly all the equations $y = T(x, m^{-1}, m)$ for all m in Σ. The construction of ab and $a + b$ is the following:

$$((a)(b) \times (0)(\infty))(1) \times C = ab, \quad ((a)(b) \times (\infty)(\infty))(0) \times C = a + b,$$

and these constructions also hold if $a = b$. T8 follows from III2; T9 from III1; T10, T11 and T12 from δIII1 and δIII2.

The converse of Theorem 1 is

THEOREM 2. *Every ternary* (S, T) *satisfying* T1 *through* T12 *coordinatizes a non-euclidean plane.*

Proof. To show the validity of I we have to consider 3 types of points whose equations are, respectively, $x = a$, $x = \infty$, $y = T(x, m, c)$, for a, m, c in S. Now, $x = a$ and $x = b$ $(a \neq b)$ are joined by the line $[\infty]$ only, and so are $x = a$ and $x = \infty$. The points $x = a$ and $y = T(x, m, c)$ are joined only by $[a, T(a, m, c)]$, and $x = \infty$ and $y = T(x, m, c)$ lie only on $[m]$. The points $y = T(x, m, c)$ and $y = T(x, n, d)$ $(m \neq n)$ are on a unique line, by T3. Finally $y = T(x, m, c)$ and $y = T(x, m, d)$ $(c \neq d)$ lie on $[m]$, and by T4 there can be no other join.

For δI we have to consider 3 types of lines: $[a, b]$, $[m]$, and $[\infty]$. The lines $[m]$ and $[n]$ $(m \neq n)$ and the lines $[m]$ and $[\infty]$ intersect at $x = \infty$ only. The lines $[m]$ and $[a, b]$ intersect at $y = T(x, m, c)$ with $b = T(a, m, c)$, which, in view of T4, determines c uniquely. The lines $[\infty]$ and $[a, b]$ meet at $x = a$, and so do $[a, b]$ and $[a, b']$ when $b \neq b'$. The lines $[a, b]$ and $[a', b']$, with $a \neq a'$, intersect at the unique point $y = T(x, m, c)$, where m and c are uniquely determined by T5 and Proposition 2 applied to the simultaneous equations $b = T(a, m, c)$ and $b' = T(a', m, c)$.

II is satisfied because the 4 points $x = 0$, $y = 0$, $x = 1$, $y = 1$ are all distinct and no 3 of them colline. Collinearity of any three of them would lead to the contradiction $0 = 1$.

Concerning III1: The tangent at $x = 0$ is $[0, 0]$. Every other line $[0, p]$ also goes through $y = T(x, p^{-1}, p)$, and $[\infty]$ also passes through $x = \infty$. No other lines contain $x = 0$. The tangent at $x = \infty$ is $[0]$. The only other lines through $x = \infty$ are of the form $[p]$ with $p \neq 0$, passing through $y = T(x, p, p^{-1})$, and $[\infty]$ passing through $x = 0$. Finally, the tangent at $y = T(x, n^{-1}, n)$, with $n \neq 0$, is $[nn, n + n]$ by T9. The line $[n^{-1}]$ is no tangent because it contains also $x = \infty$.

Concerning III2: The line $[0, b]$, $b \neq 0$, passes through 2 points of C, $x = 0$ and $y = T(x, b^{-1}, b)$. The line $[0, 0]$ passes through the single point $x = 0$. The line $[a, b]$, with $a \neq 0$, goes through $y = T(x, m^{-1}, m)$ with $b = T(a, m^{-1}, m)$, which in view of T8 yields at most 2 solutions. The line $[\infty]$ contains 2 points of C, $x = 0$ and $x = \infty$; the line $[m]$, $m \neq 0$, has 2 points, $x = \infty$ and $y = T(x, m, m^{-1})$, and the line $[0]$ only one point, $x = \infty$.

Concerning δIII1: Tangents can be only of the types $[0]$ or $[nn, n + n]$. The tangent $[0]$ passes through ∞, and no other tangent passes through ∞. Every other point on $[0]$ has an equation of the form $y = q$. Through the point $y = q$ there is exactly one tangent other than $[0]$, namely $[aa, a + a]$ with $a + a = q$. Since, in view of T10, there is exactly one such a, each point $y = q$ lies on just

2 tangents. The tangent $[nn, n+n]$ with $n \neq 0$ passes through the point $y = T(x, n^{-1}, n)$ in C. This point does not lie on $[0]$, nor can it lie on $[mm, m+m]$ with $m \neq n$, in view of T12. Every point not in C, lying on $[nn, n+n]$, is either $x = nn$ or $y = n+n$ or of the type $y = T(x, m, c)$ with $n+n = T(nn, m, c)$ and $m^{-1} \neq c$. Through $x = nn$ there is exactly one other tangent $[nn, n'+n']$ because the equation $zz = n$ $(n \neq 0)$ has, by T11, a solution $z = n' \neq n$. Through $y = n+n$ there is the second tangent $[0]$. The point $y = T(x, m, c)$ $(m \neq 0)$ lies on a tangent $[n'n', n'+n']$ other than $[nn, n+n]$ if the equation $z+z = T(zz, m, c)$ has the solution $z = n'$ besides $z = n$. But, since $m^{-1} \neq c$, this is exactly the case in view of T12. Finally, the tangent $[0, 0]$ passes through $x = 0$. Every other point on it has an equation $y = T(x, m, 0)$ with $m \neq 0$. Again $z + z = T(zz, m, 0)$, by T12, has a solution z in addition to $z = 0$, and hence another tangent exists through this point.

Concerning δIII2: The point $x = p$ can lie only on the tangents $[aa, a+a]$ with $aa = p$. By T11, there are at most 2 values of a, and therefore at most 2 tangents through $x = p$. Through $x = \infty$ there is only the tangent $[0]$. Through $y = q$ the only tangents are $[0]$ and $[aa, a+a]$, with $a + a = q$. Since, in view of T10, there is exactly one a, there are just 2 tangents. Finally, the point $y = T(x, m, c)$ $(m \neq 0)$ lies on the only tangents $[a, T(a, m, c)]$ with $T(aa, m, c) = a + a$. By T12, there are at most 2 values a, and hence at most 2 tangents. This completes the proof.

Now that the distinction between Σ and S has lost its significance, we will use S for both.

5. **Specializations of the ternary.** We will study a few instances of non-euclidean planes with special restrictions on the ternary.

Definition. We say that π_C has the *Pascal property* with respect to the line j as axis if for every 6 points P_k, Q_k $(k = 1,2,3)$ of C the following holds: If $P_1Q_2 \times P_2Q_1$ and $P_2Q_3 \times P_3Q_2$ lie on j, so does also $P_3Q_1 \times P_1Q_3$.

Theorem 3. *$(S, +)$ is an abelian group if and only if π_C has the Pascal property with respect to the axis $(\infty)(\infty)$.*

Proof. According to the construction described in the proof of Theorem 1, $(a)(b) \times (0)(a+b)$ and $(b)(c) \times (0)(b+c)$ lie on $(\infty)(\infty)$. The Pascal prooperty for the axis $(\infty)(\infty)$ holds if and only if $(a)(b+c) \times (a+b)(c)$ also lies on this axis for all choices of a, b, c in S. But this means additive associativity in S. By Proposition 16, $(S, +)$ then is an abelian group.

Theorem 4. *$(S - \{0\}, \cdot)$ is an abelian group if and only if in π_C the Pascal property with axis $(0)(\infty)$ holds.*

Proof. Again, let $a,b,c \in S$, all nonzero. The points $(a)(b) \times (1)(ab)$ and $(b)(c) \times (1)(bc)$ lie on $(0)(\infty)$. The Pascal property for the axis $(0)(\infty)$ holds if and only if $(a)(bc) \times (ab)(c)$ also lies on $(0)(\infty)$, that is, $(ab)c = a(bc)$.

THEOREM 5. *$(S, +, \cdot)$ is not necessarily left- or right-distributive, even if $(S, +)$ and $(S - \{0\}, \cdot)$ are groups.*

Proof. We employ a counter-example which we borrow from G. Pickert [7, p. 93]. Let $(S, +, \cdot)$ be the real field, and use the usual addition, but a multiplication $*$ such that

$$a * b = \begin{cases} 2ab \text{ if } a \text{ and } b \text{ are negative} \\ ab \text{ otherwise.} \end{cases}$$

Let $T(a,b,c) = a * b + c$. Then we assert that T1 through T12 are satisfied. T1, 2, 4, 6, 7, 10 and 11 are trivially fulfilled. For a proof of T3 and T5 see [7]. A verification for T8, 9 and 12 is more complicated and can be done by straightforward, though cumbersome, computation. In this example addition and multiplication are associative. However, as shown in [7], the distributive law does not hold.

On the other hand, we have obviously

THEOREM 6. *If (S,T) is linear, that is, $T(a,b,c) = ab + c$ for all a,b,c in S, and if $(S, +, \cdot)$ is a field of characteristic $\neq 2$, then* T1 *through* T12 *are valid.*

In this case Hilbert's arguments [4, appendix III] apply, and π is a projective plane over a field, in the classical sense.

THEOREM 7. *If (S,T) is linear and $(S, +, \cdot)$ a field of characteristic $\neq 2$, then for the lines $[x,y]$ of δC the equation $y^2 = 4x$ holds.*

Proof. The lines are of the form $[mm, m + m]$. Thus $y = 2m$ and $x = (y/2)^2$.

Theorem 7, in a sense, may be considered a generalization of B. Segre's theorem [8].

A final remark concerns the connection between the ternaries and M. Hall's ternary rings [3]. Comparison of their axioms yields the fact that a ternary becomes a ternary ring if $T(s,1,0) = T(1,s,0) = s$ for all s in S. This requirement corresponds to Pascal properties with axes through the points $y = T(x,s,0)$, and it assures the coordinatization of the plane dual to π by means of (S,T).

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State University of New York
at Buffalo